Q1 (a) $\mathbb{Z}_8 \oplus \mathbb{Z}_4$, elements of order 8

let $(a,b) \in \mathbb{Z}_8 \oplus \mathbb{Z}_4$

$$8 = |(a,b)| = \operatorname{lcm}(|a|,|b|)$$

$|a| = 8$ in $\mathbb{Z}_8 \Rightarrow a = 1, 3, 5, 7$

b in $\mathbb{Z}_4 \Rightarrow b = 0, 1, 2, 3$

$$(a,b) : \{(1,0), (1,1), (1,2), (1,3)$$
$$(3,0), (3,1), (3,2), (3,3)$$
$$(5,0), (5,1), (5,2), (5,3)$$
$$(7,0), (7,1), (7,2), (7,3)\}$$

which are 16 elements.

(b) $\mathbb{Z}_8 \oplus \mathbb{Z}_4$, elements of order 7

Let $(a,b) \in \mathbb{Z}_8 \oplus \mathbb{Z}_4$

$$7 = |(a,b)| = \operatorname{lcm}(|a|,|b|)$$

However, $|\mathbb{Z}_8 \oplus \mathbb{Z}_4| = 32$, and 7 doesn't divide 32,

hence, there is no element in $\mathbb{Z}_8 \oplus \mathbb{Z}_4$ of order 7.

Q1) c) $f: G \to G$ defined by $f(x) = x^2$

$f(xy) = (xy)^2 = (xy)(xy)$

$= x^2y^2$ (G is abelian)

Thus, f is homomorphtsm

Let $f(x) = e = x^2$, $x \in \ker G$

$\Rightarrow x = e$

$\Rightarrow$ The order of x is 2 $(|x| = 2)$

And we have $|x| \mid |G|$

but 2 doesn't divide 25

$\Rightarrow x = e$

Hence, f is one-to-one

Therefore, G is an isomorphism group

(d) all homomorphism and isomorphism groups on $Z_2 \oplus Z_3$

$\gcd(2,3)=1 \Rightarrow Z_2 \oplus Z_3 \approx Z_6$

- homomorphism -

$\Phi : Z_6 \to Z_6$, $Z_6 = \{0, 1, 2, 3, 4, 5\}$

$\Phi(x) = ax$, s.t, $a \in Z_6$

$\Phi(x+y) = a(x+y) = ax + ay$

$= \Phi(x) + \Phi(y)$

$\Rightarrow$ 6 homomorphisms

Now, Z_6 is cylic group, and $\exists! H$ subgroup, s.t, $|H| = 6$

Hence, only Z_6 has unique cyclic group of order 6.

Q2

a) G is an abelian group of order 35

$|G| = 35 = 5 * 7$

By Cauchy's Theorem:

5 is prime and $5 \mid 35$ $(5 \mid |G|)$

then, G has an element of order 5. #

b) Third isomorphism theorem:

~~Define the a~~ Consider the natural map $G \to G/B$.

The kernal, B, contains A. Thus, by the universal property of G/A, it follows there is a homomorphism $G/A \to G/B$.

This map is clearly surjective. In fact, it sends the left coset gA to the left coset gB.

Now suppose that gA is the kernal, then the left coset gB is the identity coset, i.e, $gB = B$, so that $g \in B$. Thus the kernal consists of those left cosets of the form gA, for $g \in B$, i.e, B/A.

The result now follows by the first Isomorphism Theorem. #

c) In order to prove $A_n \triangleleft S_n$, I need to verify $\alpha A_n \alpha^{-1} \subseteq A_n$, for any pemutation $\alpha \in S_n$.

This means, $\forall \beta \in A_n$ (i.e, even permutation), permutation $\alpha \beta \alpha^{-1}$ should also be in A_n (i.e, even permutation as well) for all possible permutations α.

First, take any $\alpha \in S_n$. ~~Similarly~~

$\Rightarrow$ The numbers of cycles of α and α^{-1} have the same parity (because $\alpha\alpha^{-1} = e$)

$\Rightarrow$ Their sum is always even.

Using: $\#\text{Cycles}(\alpha\beta\alpha^{-1}) = \#\text{Cycles}(\alpha) + \#\text{Cycles}(\beta) + \#\text{Cycles}(\alpha^{-1})$

$\Rightarrow$ the number of cycles of $\alpha\beta\alpha^{-1}$ has the same parity as number of cycles of β.

But, for the latter we know to be even, since $\beta \in A_n$.

So, it is done #

$\boxed{Q2}$ d) $H \leq G$, operation on the left cosets of H in G:

$$aHbH = abH$$

Show that $H \trianglelefteq G$

Sol: Let $x \in G$, $xHx^{-1}H = H$

$\Rightarrow xhx^{-1}H = H, \forall x \in G$ and $\forall h \in H$

$\Rightarrow xhx^{-1} \in H, \forall x \in G$ and $\forall h \in H$

$\Rightarrow H$ normal in G